

Ponderomotive Force

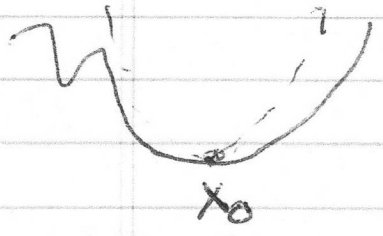


Ponderomotive Force - { opposite limit to adiabatic theory.

Recall inverted pendulum movie.

→ Motivation:

- consider motion in potential $U(x, \lambda)$.



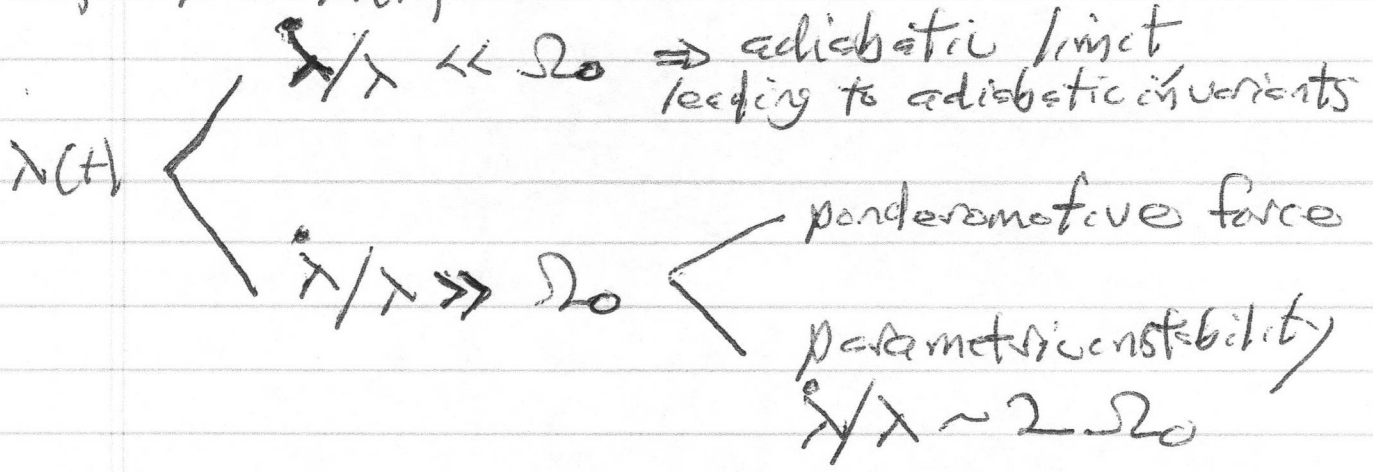
Then, particle will sit near minimum, so

$$U = U_0 + \frac{1}{2} m \Omega_0^2 (x - x_0)^2$$

⇒ motion has natural frequency Ω_0

$$U'' = \frac{1}{2} m \Omega_0^2 \Big|_{x_0}$$

- now, $\lambda = \lambda(t)$



- consider 1D problem with high frequency driver F

→ high frequency drive

c.e. $m \ddot{x} = - \frac{dU}{dx} + F$
"small"

- Recall movie of inverted pendulum:
- what happened?
 - why is it stable while inverted?
 - is there a criticality condition?
what sets it?
 - How to approach theoretically?
 - what leverage (small parameter) do we have on the problem?

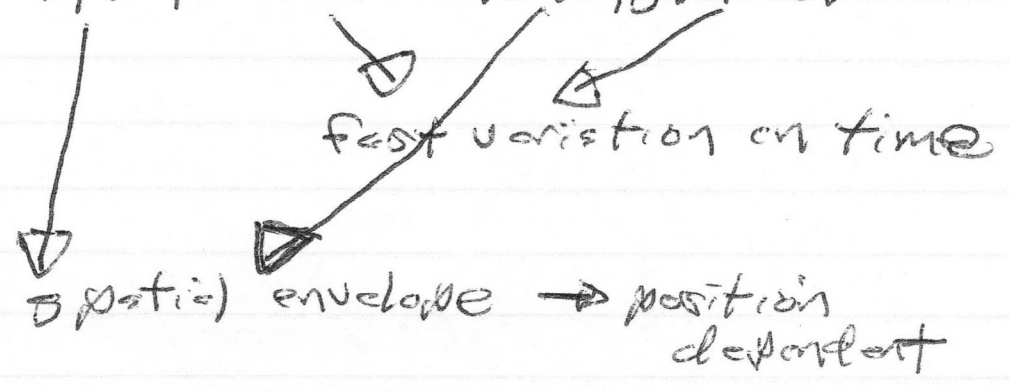
N.B. See "inverted pendulum"
videos on YouTube.

idea driver on the pendulum

now $F = F_1 \cos \omega t + F_2 \sin \omega t$

$= F_1(x) \cos \omega t + F_2(x) \sin \omega t$

$\omega \gg \Omega_0$



→ idea is that F_1, F_2 induce oscillation on motion in U .

Can we describe "mean motion"?

⇒ Primary interest in centroid motion...

or

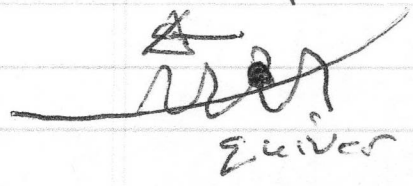
$x(t) = y(t) + \epsilon(t)$

↓
slow, centroid motion

↪ fast, quivering motion on frequency ω
[quiver]

[quiver center]

centroid



Now, $x = y + \epsilon$ separates motion into slow, fast components

$y \rightarrow$ slow
 $\epsilon \rightarrow$ fast

* Time/Frequency scale separation is key.

Now, since ϵ tracks F :

$$\tau = 2\pi/\omega ; \quad \frac{1}{\tau} \int_0^{\tau} dt \epsilon(t) = 0.$$

↓
period average of high frequency oscillation vanishes.

so, EOM:

$$m \ddot{x} = -\frac{dU}{dx} + F$$

$$\Rightarrow \text{and } x = y + \epsilon$$

$$m(\ddot{y} + \ddot{\epsilon}) = -\frac{d}{dx}(U(y+\epsilon)) + F(y+\epsilon)$$

now expand:

(assumes ϵ small, no resonances)

$$m(\ddot{y} + \ddot{\epsilon}) = -\frac{dU(x)}{dx} - \epsilon \frac{d^2 \psi}{dx^2} + F(y) + \epsilon \frac{dF}{dx}$$

note this is a best

$$\epsilon \sim \cos \omega t$$

$$dF/dx \sim \cos \omega t$$

⇒ generates low frequency!

⇒

$$m(\ddot{y} + \ddot{\epsilon}) = -\frac{dU(y)}{dy} - \epsilon \frac{d^2 u(y)}{dy^2} + F(y) + \epsilon \frac{dF}{dy}$$

- ① → slow time scale
- ② → fast time scale
- ③ → fast time scale ($\sim \cos \omega t$)
- ④ → slow (via best)

Now, to separate:

$$\langle \gamma \rangle = \frac{t}{T} \int_0^T dt$$

So

$$m \langle \ddot{y} \rangle + m \langle \ddot{\xi} \rangle = - \left\langle \frac{dU}{dy} \right\rangle - \left\langle \xi \frac{d^2 U}{dy^2} \right\rangle + \langle F \rangle + \left\langle \xi \frac{\partial F}{\partial y} \right\rangle$$

$$\langle \ddot{\xi} \rangle = 0$$

$$\langle \xi \rangle = 0$$

$$m \ddot{y} = - \frac{dU}{dy} + \left\langle \xi \frac{\partial F}{\partial y} \right\rangle$$

\uparrow
 non-trivial
 solution

slow
or mean
field,
eg.

For fast,

$$m \ddot{\xi} = - \xi \frac{d^2 U}{dy^2} + F(y)$$

but: $m \ddot{\tilde{\epsilon}} \sim m \omega^2 \tilde{\epsilon}$

$$\tilde{\epsilon} \frac{d^2 y}{dy^2} \sim m \Omega_0^2 \tilde{\epsilon}$$

and $\omega^2 \gg \Omega_0^2$

So

$m \ddot{\tilde{\epsilon}} = f(y)$

 \rightarrow fast eqn.

Proceed:

- (a) - solve fast eqn.
- (b) - iterate into slow \Rightarrow calculate best
- (c) - solve slow

Now;

(a) $m \ddot{\tilde{\epsilon}} = f_1(y) \cos \omega t + f_2(y) \sin \omega t$

$$\tilde{\epsilon} = \frac{-f(y)}{m\omega^2} \rightarrow \text{fast given variation}$$

(f is space, time dependent; time dependence is slow)

A)

then, for best:

$$\text{best} = \langle \bar{y} \rangle = \left\langle \bar{y} \frac{\partial F}{\partial y} \right\rangle$$

$$= \left\langle \frac{-F}{m\omega^2} \frac{\partial F}{\partial y} \right\rangle$$

⇒

$$m\ddot{y} = -\frac{dU}{dy} = -\frac{1}{m\omega^2} \left\langle F \frac{\partial F}{\partial y} \right\rangle$$

here:

$$F = F_1 \cos \omega t + F_2 \sin \omega t$$

$$\langle \rangle = \frac{1}{T} \int_0^T dt$$

⇒

$$m\ddot{y} = -\frac{dU}{dy} = -\frac{1}{2m\omega^2} \left\langle F_1 \frac{\partial F_1}{\partial y} + F_2 \frac{\partial F_2}{\partial y} \right\rangle$$

(cross terms gone)

$$m\ddot{y} = -\frac{dU}{dy} = -\frac{1}{2m\omega^2} \frac{d}{dy} \langle F^2 \rangle$$

$$\{F^2\} = \frac{1}{2} (f_1^2 + f_2^2)$$

\Rightarrow (C)

$$m\ddot{y} = -\frac{d}{dy} \left[U(y) + \frac{1}{4m\omega^2} (f_1^2(y) + f_2^2(y)) \right]$$

EOM for $y \Leftrightarrow$ mean field eqn.

Effective potential becomes:

$$U_{\text{eff}} = U(y) + \frac{1}{4m\omega^2} (f_1^2(y) + f_2^2(y))$$

where ponderomotive potential is:

$$U_{\text{pondero}} = \frac{1}{4m\omega^2} (f_1^2 + f_2^2)$$

is piece induced by high frequency quiver.

then,

$$F_{\text{ponderomotive}} = -\nabla U_{\text{pondero}}$$

$$= -\frac{1}{4m\omega^2} \frac{d}{dy} (F_1^2 + F_2^2)$$

- ponderomotive force
- ~ gradient of optical envelope of high frequency energy field
- observe:

$$\frac{1}{4m\omega^2} (F_1^2 + F_2^2) = \frac{1}{2} m \langle \dot{\xi}^2 \rangle$$

↓
Zwiler kinetic energy

alternatively,

$$U_{\text{ponderomotive}} = \frac{1}{2} m \langle \dot{\xi}^2 \rangle$$

where:

$$m \ddot{\xi} = f$$

→ effective kinetic energy field of quiver kinetic energy

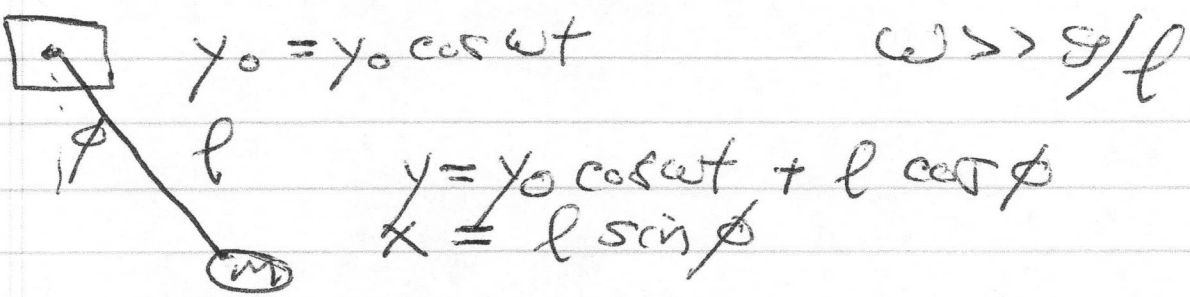
→ gradient in kinetic energy field induces mean force

Note: Wave momentum theorem

$$\frac{dP_w}{dt} = -\nabla H = -\nabla \epsilon \quad (iD) \quad (\partial_t \epsilon = 0)$$
$$\rightarrow -\nabla \cdot \underline{I}$$

has character of ponderomotive.
Gradient of energy density exerts force!

→ (Classic) Example - Inverted Pendulum (recall movie)



Consider "the usual" pendulum with vertically oscillating point of support.

$$L = T - U$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgl(1 - \cos\phi)$$

$$L = \frac{1}{2} m \left(l^2 \cos^2\phi \dot{\phi}^2 + (\omega y_0 \sin\omega t + l \sin\phi \dot{\phi})^2 \right) - mgl(1 - \cos\phi)$$

$$= \frac{1}{2} m \left(l^2 \cos^2\phi + l^2 \sin^2\phi \right) \dot{\phi}^2$$

$$+ 2\omega y_0 l \dot{\phi} \sin\omega t \sin\phi + \omega^2 y_0^2 \sin^2\omega t$$

h.c. on y_0
 \Rightarrow weak
 (ϕ small)

$$L = \frac{1}{2} m \left(l^2 \dot{\phi}^2 + 2\omega y_0 l \dot{\phi} \sin\phi \sin\omega t \right) + mgl \cos\phi$$

oscillating term

LEOM

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$

⇒

$$\frac{d}{dt} \left(m l^2 \dot{\phi} + m \omega y_0 l \sin \omega t \sin \phi \right)$$

$$= -m g l \sin \phi + m \omega y_0 l \dot{\phi} \sin \omega t \cos \phi$$

⇒

$$m l^2 \ddot{\phi} + m \omega y_0 l (\omega \cos \omega t \sin \phi + \sin \omega t \cos \phi \dot{\phi})$$

$$= -m g l \sin \phi + m \omega y_0 l \dot{\phi} \sin \omega t \cos \phi$$

$$m l^2 \ddot{\phi} = -m g l \sin \phi - m \omega^2 y_0 l \sin \phi \cos \omega t$$

$$\ddot{\phi} = -\frac{g}{l} \sin \phi - \frac{\omega^2 y_0}{l} \sin \phi \cos \omega t$$

in form:

$$\ddot{\phi} = -\frac{dU(\phi)}{d\phi} + f_1(\phi) \cos \omega t$$

\Leftrightarrow

$$\ddot{\phi} = -\frac{g}{l} \sin \phi - \frac{\omega^2 y_0}{l} \sin \phi \cos \omega t$$

\approx

$$U(\phi) = \frac{g}{l} \cos \phi$$

$$\omega \gg \sqrt{g/l}$$

$$f_1(\phi) = -\frac{\omega^2 y_0}{l} \sin \phi$$

To solve:

$$\phi = \langle \phi \rangle + \tilde{\phi}$$

$$= \bar{\phi} + \tilde{\phi}$$

notational brevity

$$\ddot{\bar{\phi}} + \ddot{\tilde{\phi}} = -\frac{g}{l} (\sin[\bar{\phi} + \tilde{\phi}]) - \frac{\omega^2 y_0}{l} \sin(\bar{\phi} + \tilde{\phi}) \cos \omega t$$

expanding \Rightarrow

$$\ddot{\bar{\phi}} + \ddot{\tilde{\phi}} = \overset{\textcircled{1}}{-\frac{g}{l} \sin \bar{\phi}} - \overset{\textcircled{2}}{\tilde{\phi} \frac{g}{l} \cos \bar{\phi}}$$

$$- \overset{\textcircled{3}}{\omega^2 \frac{y_0}{l} \sin \bar{\phi} \cos \omega t} - \overset{\textcircled{4}}{\omega^2 \frac{y_0}{l} \cos \bar{\phi} \tilde{\phi} \cos \omega t}$$

then average:

$$\langle \textcircled{2} \rangle = \langle \textcircled{3} \rangle = 0$$

$$\ddot{\bar{\phi}} = -\frac{g}{l} \sin \bar{\phi} - \omega^2 \frac{y_0}{l} \cos \bar{\phi} \langle \tilde{\phi} \cos \omega t \rangle$$

To compute $\langle \tilde{\phi} \rangle$, need $\tilde{\phi}$

$$\ddot{\tilde{\phi}} = -\tilde{\phi} \frac{g}{l \cos \bar{\phi}} - \omega^2 \frac{y_0}{l} \sin \bar{\phi} \cos \omega t$$

$g/l \ll \omega^2$

$$\tilde{\phi} = a(\bar{\phi}) \cos \omega t$$

$$-\omega^2 a(\bar{\phi}) \cos \omega t = -\omega^2 \frac{y_0}{l} \sin \bar{\phi} \cos \omega t$$

$$a(\bar{\phi}) = \frac{y_0}{l} \sin \bar{\phi}$$

$$\begin{aligned} \langle \bar{\phi} \cos \omega t \rangle &= \left\langle \frac{y_0}{l} \sin \bar{\phi} \cos^2 \omega t \right\rangle \\ &= \frac{y_0}{2l} \sin \bar{\phi} \end{aligned}$$

so

$$\begin{aligned} \langle \dot{\phi} \rangle &= -\frac{\omega^2 y_0^2}{2l^2} \sin \bar{\phi} \cos \bar{\phi} \\ &= -\frac{\omega^2 y_0^2}{4l^2} \sin 2\bar{\phi} \end{aligned}$$

so \Rightarrow

$$\ddot{\bar{\phi}} = -\frac{g}{l} \sin \bar{\phi} - \frac{\omega^2 y_0^2}{4l^2} \sin 2\bar{\phi}$$

$$-\omega^2 a(\bar{\phi}) \cos \omega t = -\omega^2 \frac{y_0}{l} \sin \bar{\phi} \cos \omega t$$

$$a(\bar{\phi}) = \frac{y_0}{l} \sin \bar{\phi}$$

$$\langle \ddot{\phi} \cos \omega t \rangle = \frac{y_0}{l} \langle \sin \bar{\phi} \cos^2 \omega t \rangle$$

$$= \frac{y_0}{2l} \sin \bar{\phi}$$

so

$$\langle \textcircled{4} \rangle = -\frac{\omega^2 y_0^2}{2l^2} \sin \bar{\phi} \cos \bar{\phi}$$

$$= -\frac{\omega^2 y_0^2}{4l^2} \frac{d}{d\bar{\phi}} \sin^2 \bar{\phi}$$

so

$$\ddot{\bar{\phi}} = -\frac{d}{d\bar{\phi}} \left(-\frac{g}{l} \cos \bar{\phi} + \frac{\omega^2 y_0^2}{4l^2} \sin^2 \bar{\phi} \right)$$

$$U_{\text{eff}} = U_0 + U_{\text{pend.}}$$

$$U_0 = -\frac{g}{l} \cos \bar{\phi}$$

effective potential.

$$U_{\text{pend.}} = \frac{\omega^2 y_0^2}{4l^2} \sin^2 \bar{\phi}$$

Now, with inverted pendulum in mind,
look for minimum (1D)

$$U(\bar{\phi}) = -\frac{g}{l} \cos \bar{\phi} + \frac{\omega^2 y_0^2}{4l^2} \sin^2 \bar{\phi}$$

$$= -\frac{g}{l} \cos \bar{\phi} + \frac{\omega^2 y_0^2}{4l^2} \left(\frac{1}{2} - \frac{1}{2} \cos 2\bar{\phi} \right)$$

extrema:

$$\frac{dU}{d\bar{\phi}} = \frac{g}{l} \sin \bar{\phi} + \frac{\omega^2 y_0^2}{4l^2} \sin 2\bar{\phi} = 0$$

$$\bar{\phi} = 0 \quad \text{is minimum}$$

$$\bar{\phi} = \pi \quad \text{is extremum.}$$

Now,

$$\frac{d^2U}{d\bar{\phi}^2} = \frac{g}{l} \cos \bar{\phi} + \frac{\omega^2 y_0^2}{2l^2} \cos 2\bar{\phi}$$

for $\bar{\phi} = \pi$

$$= -\frac{g}{l} + \frac{\omega^2 y_0^2}{2l^2}$$

\Rightarrow extremum can be minimum for $\frac{\omega^2 y_0^2}{2l^2} > g/l$.

So for $\frac{\omega^2 y_0^2}{2l^2} > \frac{g}{l}$, $\bar{\phi} = \pi$ is a (stable) minimum.

\Rightarrow

- ~~pendulum~~ pendulum effective potential has second minimum at inversion ($\bar{\phi} = \pi$) for $\frac{\omega^2 y_0^2}{2l^2} > \frac{g}{l}$

- establishes critical ponderomotive potential strength.

Comments:

\rightarrow have encountered ponderomotive force

gradient on
c.e. Aspatially varying envelope of
high frequency energy field

→ can think of ponderomotive potential
as effective radiation pressure.

∴ How do EM waves exert
a ponderomotive force?

Hint: Analogy with acoustic
streaming?